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CALIFORNIA UNIV SAN DIEGO LA JOLLA DEPT OF APPLIED P--ETC F/G 12/1
DISCRETE-TIME SPECTRAL ESTIMATION OF CONTINUOUS-TIME PROCESSES --ETC(U)
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DISCRETE-TIME SPECTRAL ESTIMATION OF CONTINUOUS-TIME PROCESSES -
THE ORTHOGONAL SERIES METHOD

(Spectral Estimation by Orthogonal Series)

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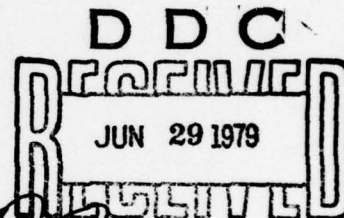
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This research was supported by the Office of Naval Research under Contract

~~NO~~ 14-75-C-0652

AMS 1970 subject classifications. Primary 62M15.

Key words and phrases. Spectral estimation, orthogonal series estimates, convergence rates.

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Summary. Discrete-Time Spectral Estimation of Continuous-Time Processes -
The Orthogonal Series Method.

Let $\{X(t), -\infty < t < \infty\}$ be a stationary time series with spectral density function $\phi(\lambda)$. Let $\{t_n\}$ be a stationary Poisson point process on the real line. The existence of consistent estimates of $\phi(\lambda)$ based on the discrete-time observations $\{X(t_n)\}_{n=1}^N$, when the actual sampling times are not known, has been an open question (Beutler (1970)). Using an orthogonal series method, a class of spectral estimates is considered and its uniform and integratedly uniform consistency in quadratic mean is established. Rates of convergence are established and are compared with the optimal rates of the available (Brillinger (1972), Masry (1978)) kernel-type estimates based on the observations $\{X(t_n), t_n\}_{n=1}^N$.

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1. Introduction. There is an extensive literature on the subject of spectral estimation of continuous and discrete parameters time series. Here we are concerned with the estimation of the spectral density function $\phi(\lambda)$ of a time series $X = \{X(t), -\infty < t < \infty\}$ based on the discrete-time observations $\{X(t_n), t_n\}$, where the sampling process $\{t_n\}$ constitutes a stationary point process on the real line. Brillinger (1972), in his fundamental work on the spectral analysis of stationary interval functions, discussed the consistency and asymptotic normality as $T \rightarrow \infty$ of kernel-type spectral estimates $\phi_T(\lambda)$ based on the modified periodogram

$$(1.1) \quad I_T(\lambda) = \frac{1}{2\pi T} \left\{ \left| \int_0^T e^{-it\lambda} X(t) dN(t) \right|^2 - \int_0^T X^2(t) dN(t) \right\}$$

where $N(t) = N((0, t])$ is the counting process associated with $\{t_n\}$. Considering $\phi_T(\lambda)$ as an estimate based on discrete-time observations, it clearly employs a random sample size $N(T)$. Assuming a deterministic sample size N , kernel-type spectral estimates $\hat{\psi}_N(\lambda)$ based on the observations $\{X(t_n), t_n\}_{n=1}^N$, where N is a positive integer and $\{t_n\}$ is a stationary Poisson point process, have been considered by Masry (1978) where their pointwise consistency in quadratic mean as $N \rightarrow \infty$ is derived; $\hat{\psi}_N(\lambda)$ is based on the modified periodogram

$$(1.2) \quad J_N(\lambda) = \frac{1}{2\pi N} \left\{ \left| \sum_{n=1}^N e^{-it_n\lambda} X(t_n) \right|^2 - \sum_{n=1}^N X^2(t_n) \right\}.$$

For a Poisson sampling process $\{t_n\}$, the second order statistics of $J_N(\lambda)$ and $I_T(\lambda)$ are distinct but the second order statistics of the corresponding kernel-type estimates $\hat{\psi}_N(\lambda)$ and $\phi_T(\lambda)$ are virtually identical, as shown in [5].

The above estimates require the knowledge of the actual values of the sampling instants $\{t_n\}_{n=1}^N$ (to evaluate $e^{it_n\lambda}$) in addition to the data sequence

$\{X(t_n)\}_{n=1}^N$. This paper investigates the estimation of $\phi(\lambda)$ on the basis of the data sequence $\{X(t_n)\}_{n=1}^N$ alone, i.e., when the actual sampling instants are not known. (Such a situation may arise in randomly-sampled data systems where the actual sampling times are not transmitted. See also Gaster and Roberts (1975) for examples in fluid mechanics.) Clearly, the periodogram approach (1.1)-(1.2) is no longer applicable. In fact, the existence of consistent estimates of $\phi(\lambda)$ under these circumstances has been an open question as noted in Beutler (1970). This paper resolves this question when the sampling process $\{t_n\}$ is Poisson: By employing an orthogonal series method, a class of spectral estimates $\hat{\phi}_N(\lambda)$ is introduced and its uniform consistency and integrated uniform consistency in quadratic mean is established. The approach is similar to the one used by Čencov (1962), Schwartz (1967), Watson (1969) and Rosenblatt (1971) for probability density estimation. However, unlike these works, the complete orthogonal set in L_2 used for the representation of $\phi(\lambda)$, cannot be arbitrary and is, in fact, generated by the statistics of the sampling process $\{t_n\}$.

In Section 2, the series representation for $\phi(\lambda)$ is introduced and some preliminary results are given. Conditions for the consistency of $\hat{\phi}_N(\lambda)$ and bounds on the two types of errors are established in Section 3. These results are compared in Section 4 with the optimal convergence properties of the estimates based on the periodogram (1.1) or (1.2). The comparison indicates that the orthogonal series estimate $\hat{\phi}_N(\lambda)$ has appreciably slower rates of convergence. The proofs are collected in Section 5. The question of whether these rates are the best possible remains open.

2. Preliminaries. Throughout this paper, $X = \{X(t), -\infty < t < \infty\}$ is a real stationary measurable fourth order process with mean zero, continuous covariance function $C(t) \in L_1$, spectral density $\phi(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} C(t) \exp[-it\lambda] dt$ and fourth order cumulant function $Q(u_1, u_2, u_3, \cdot)$. The sampling process $\{t_n\}$ is assumed to

be a stationary Poisson point process on $[0, \infty)$, independent of X , with known mean intensity β , i.e., $t_0 = 0$ a.s., $t_n = t_{n-1} + T_n$, $n = 1, 2, \dots$, where the T_n 's are independent identically distributed random variables with exponential distribution $F(x) = [1 - \exp(-\beta x)]$.

The basic idea of the paper is to employ an appropriate orthogonal series representation for $\phi(\lambda)$ whose coefficients can be estimated from the data sequence $\{X(t_n)\}_{n=1}^N$ alone. Note that if $\{U_n(\lambda)\}_{n=1}^\infty$ is an arbitrary complete orthonormal set in $L_2(-\infty, \infty)$, then $\phi(\lambda) = \sum_{n=1}^\infty b_n U_n(\lambda)$ in L_2 , $b_n = \int_{-\infty}^\infty \phi(\lambda) U_n^*(\lambda) d\lambda$ and, as in probability density estimation by orthogonal series [3][9][12], one could estimate b_n by $\hat{b}_n(N) = (1/\beta) \int_{-\infty}^\infty J_N(\lambda) U_n^*(\lambda) d\lambda$, where $J_N(\lambda)$ is given in (1.2). Then

$$\hat{b}_n(N) = (1/2\pi\beta N) \sum_{\substack{j,k=1 \\ j \neq k}}^N X(t_j) X(t_k) u_n^*(t_j - t_k)$$

where $u_n(t) = \int_{-\infty}^\infty \exp[it\lambda] U_n(\lambda) d\lambda$. Thus, $\hat{b}_n(N)$ requires the knowledge of the sampling instants $\{t_n\}_{n=1}^N$ in addition to the data sequence $\{X(t_k)\}_{k=1}^N$. Hence, the basis $\{U_n(\lambda)\}$ cannot be arbitrary for our purposes. Next we note (Shapiro and Silverman (1960)) that the discrete-parameter process $\{X(t_k)\}$ has mean zero and covariance sequence $\{c_n\}$

$$(2.1) \quad c_n = E[X(t_{k+n})X(t_k)] = \int_0^\infty C(t) f_n(t) dt, \quad n = 1, 2, \dots$$

where $f_n(t) = \beta[(\beta t)^{n-1}/(n-1)!] \exp(-\beta t)$, $n = 1, 2, \dots$. The set $\{f_n(t)\}$ is complete in $L_2(0, \infty)$ and its orthonormalization yields

$$(2.2) \quad g_n(t) = \sqrt{2\beta} L_{n-1}(2\beta t) e^{-\beta t} 1_{[0, \infty)}(t), \quad n = 1, 2, \dots,$$

where $L_n(t)$ is the n^{th} Laguerre polynomial. Note that [10] $g_n(t) = \sum_{k=1}^n \theta_{n,k} f_k(t)$

where

$$(2.3) \quad \theta_{n,k} = \sqrt{2/\beta} (-2)^{k-1} \binom{n-1}{k-1}.$$

Since $C(t) \in L_2(-\infty, \infty)$, we have the L_2 expansion $C(t) = \sum_{n=1}^{\infty} a_n g_n(|t|)$ where

$$(2.4) \quad a_n = \int_0^{\infty} C(t) g_n(t) dt = \sum_{k=1}^n \theta_{n,k} c_k.$$

Hence

$$(2.5) \quad \phi(\lambda) = \sum_{n=1}^{\infty} a_n G_n(\lambda)$$

in $L_2(-\infty, \infty)$ where, by direct Fourier transform of (2.2),

$$(2.6) \quad G_n(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} g_n(|t|) dt = -\frac{\sqrt{2\beta}}{\pi} \frac{\cos[(2n-1)\tan^{-1}(\lambda/\beta)]}{\sqrt{\lambda^2 + \beta^2}}, \quad n = 1, 2, \dots$$

$\{G_n(\lambda)\}_{n=1}^{\infty}$ is complete and orthogonal in $L_2(-\infty, \infty)$ with respect to even functions on $(-\infty, \infty)$. The series expansion (2.5) has been considered by Shapiro and Silverman (1960) in connection with a concept of "alias-free" sampling.

The approach: Given the data sequence $\{X(t_n)\}_{n=1}^N$ estimate c_n by

$$(2.7) \quad \hat{c}_n(N) = \begin{cases} \frac{1}{N} \sum_{k=1}^{N-n} X(t_{k+n}) X(t_k), & 1 \leq n < N \\ 0, & N \leq n \end{cases}$$

Then, via (2.4), estimate the expansion coefficient a_n by

$$(2.8) \quad \hat{a}_n(N) = \sum_{k=1}^n \theta_{n,k} \hat{c}_k(N),$$

and finally, estimate $\phi(\lambda)$ by

$$(2.9) \quad \hat{\phi}_N(\lambda) = \sum_{n=1}^{\infty} \gamma_n(N) \hat{a}_n(N) G_n(\lambda)$$

where $\{\gamma_n(N)\}$ is an appropriate averaging sequence to be specified below.

A bound on the rate of decay of $\{a_n\}$ is needed and is given below;
 $AC^r[0, \infty)$ denotes the set of functions which are r -times absolutely continuous
on $[0, \infty)$.

LEMMA 2.1. For $t \geq 0$, let $C(t) \in AC^{r-1}[0, \infty)$ such that

$$(2.10) \quad t^{r/2} C^{(k)}(t) \in L_2(0, \infty) \text{ for } k = 0, 1, \dots, r.$$

Then for $n = 1, 2, \dots$

$$|a_n| \leq A_1(r) n^{-r/2}$$

where

$$A_1(r) = \frac{1}{\sqrt{2\beta}} \left\| t^{r/2} e^{t/2} \frac{d^r}{dt^r} [C(t/2\beta) e^{-t/2}] \right\|_{L_2(0, \infty)}.$$

REMARK. The hypothesis of Lemma 2.1 does not require the differentiability
of $C(t)$ at the origin. Thus, the spectral moments of $\phi(\lambda)$ need not exist. For
example, $C(t) = e^{-|t|}$ satisfies the hypothesis of Lemma 2.1 for every integer $r \geq 1$.
Similarly, if $\phi(\lambda)$ is a rational function in λ , then Lemma 2.1 holds for every
integer $r \geq 1$.

In investigating the convergence in quadratic mean of the estimate (2.9)
for a fixed λ , the pointwise convergence of the L_2 expansion (2.5) is needed.

We have

LEMMA 2.2. Assume $C(t)$ satisfies the hypothesis of Lemma 2.1 for some
 $r > 2$. Then

$$(2.11) \quad \phi(\lambda) = \sum_{n=1}^{\infty} a_n G_n(\lambda)$$

uniformly on $(-\infty, \infty)$.

3. Consistency and rates of convergence. We first consider the consistency of the estimates $\hat{c}_n(N)$ and $\hat{a}_n(N)$. The following assumption is needed.

ASSUMPTION 3.1. The process X satisfies

$$(i) \quad tC(t) \in L_2(-\infty, \infty)$$

$$(ii) \quad |Q(u_1, u_2, u_3)| \leq h(u_1, u_2, u_3)$$

where h is even and nonincreasing on $[0, \infty)$ in each variable such that

$$\int_{-\infty}^{\infty} h(0, u, 0) du < \infty.$$

THEOREM 3.1. Under Assumption 3.1, the estimate $\hat{c}_n(N)$ is consistent in quadratic mean with

$$(i) \quad E[\hat{c}_n(N)] = (1 - \frac{n}{N})c_n$$

$$(ii) \quad \text{Var}[\hat{c}_n(N)] \leq A_2/N$$

where A_2 is a constant independent of n and N .

Before considering the consistency of the estimate $\hat{a}_n(N)$, we note that the mapping (2.4) of $\{c_n\}$ to $\{a_n\}$ is unbounded in ℓ_∞ since $\sum_{k=1}^n \theta_{n,k} = \sqrt{2/\beta}(-1)^{n-1}$ but $\sum_{k=1}^n |\theta_{n,k}| = \sqrt{2/\beta} 3^{n-1}$. Thus, since $\text{Var}[\hat{a}_n(N)] = \sum_{k,\ell=1}^n \theta_{n,k} \theta_{n,\ell} \text{cov}[\hat{c}_k(N), \hat{c}_\ell(N)]$, a small variability in the estimate $\hat{c}_n(N)$ is likely to produce a large variability in the estimate $\hat{a}_n(N)$ for large n . A simulation study in [4] appears to confirm this observation. We have

THEOREM 3.2. Under Assumption 3.1 the estimate $\hat{a}_n(N)$ is consistent in quadratic mean with

$$(i) \quad E[\hat{a}_n(N)] = a_n - (1/N)[n a_n - (n-1)a_{n-1}]$$

$$(ii) \quad \text{Var}[\hat{a}_n(N)] \leq A_3 \frac{3^{2n}}{N}$$

where $A_3 = 2A_2/9\beta$.

Consider now the class of spectral estimates $\hat{\phi}_N(\lambda)$ defined by (2.9), where the averaging sequence $\{\gamma_n(N)\}$ is of the form

$$(3.1) \quad \gamma_n(N) = h(e^{\alpha n}/N^b), \quad \alpha > \ln 3, \quad 0 < b < \frac{\alpha}{2 \ln 3}$$

and $h(u)$ is a real even function on the real line satisfying

$$(3.2) \quad \begin{aligned} (i) \quad & |h(u)| \leq h(0) = 1 \quad \text{for all } u, \\ (ii) \quad & |uh(u)| \leq K_1 \quad \text{for } |u| \geq 1, \\ (iii) \quad & 1-h(u) \leq K_2|u| \quad \text{for all } u. \end{aligned}$$

The kernel $\{\gamma_n(N)\}$ is then of the exponential type as defined by Parzen (1958). The choice of exponential-type kernel (rather than algebraic) and the conditions on the parameters α and b are necessitated by the variance expression of $\hat{a}_n(N)$.

THEOREM 3.3. Assume X satisfies Assumption 3.1 and the hypothesis of Lemma 2.1 for some $r > 1$. Then the estimate (2.9) is integratedly uniformly consistent in quadratic mean with

$$E \int_{-\infty}^{\infty} |\hat{\phi}_N(\lambda) - \phi(\lambda)|^2 d\lambda \leq B_1 [1/\ln N]^{r-1} (1+o(1))$$

where

$$B_1 = 4 A_1^2(r) (2\alpha/b)^{r-1} / (r-1).$$

Next we have

THEOREM 3.4. Assume X satisfies Assumption 3.1 and the hypothesis of Lemma 2.1 for some $r > 2$. Then the estimate (2.9) is uniformly consistent in quadratic mean with

$$E |\hat{\phi}_N(\lambda) - \phi(\lambda)|^2 \leq B_2 [1/\ln N]^{r-2} (1+o(1))$$

uniformly in λ , where

$$B_2 = (32/\pi^2 B) A_1^2(r) (2\alpha/b)^{r-2} (r-2)^{-2}.$$

4. Discussion. We first establish the optimal convergence properties of the kernel-type estimate $\hat{\psi}_N(\lambda)$ based on the periodogram (1.2). $\hat{\psi}_N(\lambda)$ is of the form [2][5]

$$(4.1) \quad \hat{\psi}_N(\lambda) = (1/\beta) \int_{-\infty}^{\infty} W_N(\lambda-\mu) J_N(\mu) d\mu$$

where $W_N(\lambda) = M_N K(M_N \lambda)$, $M_N \rightarrow \infty$ and $M_N/N \rightarrow 0$ as $N \rightarrow \infty$ and the kernel $K(\lambda)$ is a real even continuously differentiable function on $(-\infty, \infty)$ satisfying

$$(4.2) \quad \begin{array}{ll} (i) \quad \sup_{-\infty < \lambda < \infty} |K(\lambda)| < \infty & (ii) \quad \sup_{-\infty < \lambda < \infty} |K'(\lambda)| < \infty \\ (iii) \quad \int_{-\infty}^{\infty} |K(\lambda)| d\lambda < \infty & (iv) \quad \int_{-\infty}^{\infty} K(\lambda) d\lambda = 1. \end{array}$$

Aside from the conditions on the fourth order cumulant $Q(u_1, u_2, u_3)$ we have by [5], Theorems 1 and 3 (or [2], Theorem 4.3), that if $tC(t) \in L_1(-\infty, \infty)$ then

$$E[\hat{\psi}_N(\lambda)] = \int_{-\infty}^{\infty} W_N(\lambda-\mu) \phi(\mu) d\mu + o(1/N)$$

$$\text{Var}[\hat{\psi}_N(\lambda)] = 2\pi\beta(M_N/N) \left[\phi(\lambda) + \frac{C(0)}{2\pi\beta} \right]^2 (1 + \delta_{0,\lambda}) \int_{-\infty}^{\infty} K^2(u) du (1 + o(M_N/N)).$$

To find the optimal convergence properties of $\hat{\psi}_N(\lambda)$ we proceed as in Wahba (1975). Assume that for some integer $r \geq 1$, $K(\lambda)$ satisfies in addition

$$(4.2) \quad \begin{array}{l} (v) \quad \int_{-\infty}^{\infty} \lambda^j K(\lambda) d\lambda = 0, \quad j = 1, \dots, r-1 \\ (vi) \quad \int_{-\infty}^{\infty} |\lambda|^r |K(\lambda)| d\lambda < \infty \end{array}$$

and that

$$(4.3) \quad t^r C(t) \in L_1(-\infty, \infty).$$

Expanding $\phi(\lambda+\mu)$ in a Taylor series, and using (4.2)(iv)-(vi) in

$$\int_{-\infty}^{\infty} K(-u)\phi(\lambda+u/M_N)du, \text{ we find } b^2[\hat{\psi}_N(\lambda)] \leq B_3 M_N^{-2r}(1 + o(1))$$

where

$$B_3 = \frac{1}{[2\pi r!]^2} \left[\int_{-\infty}^{\infty} |u|^r |K(u)| du \right]^2 \left[\int_{-\infty}^{\infty} |t|^r |C(t)| dt \right]^2.$$

Thus, ignoring a factor $(1 + o(M_N/N))$ in the variance and $(1 + o(1))$ in the bias, we have

$$(4.4) \quad E|\hat{\psi}_N(\lambda) - \phi(\lambda)|^2 \leq B_3 M_N^{-2r} + B_4(M_N/N)$$

where

$$B_4 = 4\pi\beta \left[\int_{-\infty}^{\infty} |C(t)| dt + \frac{C(0)}{2\pi\beta} \right]^2 \int_{-\infty}^{\infty} K^2(u) du.$$

Then choosing $M_N = [2rB_3/B_4]^{1/(2r+1)} N^{1/(2r+1)}$, which minimizes the right-hand side of (4.4), we have finally

$$(4.5) \quad E|\hat{\psi}_N(\lambda) - \phi(\lambda)|^2 \leq B_5 N^{-2r/(2r+1)}(1 + o(1))$$

where

$$B_5 = \frac{(2r+1)}{(2r)^{2r/(2r+1)}} [B_3 B_4^{2r}]^{1/(2r+1)}.$$

The parameter r of (4.3) and Lemma 2.1 represents a degree of smoothness of $\phi(\lambda)$. For the same value of r , the hypothesis of Lemma 2.1 is generally more restrictive than (4.3), even though (4.3) implies (2.10) with $k = 0$. For the same value of r , the mean square error of the estimate $\hat{\psi}_N(\lambda)$ is $O(1/N^{2r/(2r+1)})$ whereas for the series estimate $\hat{\phi}_N(\lambda)$ it is $O(1/[\ln N]^{r-2})$ by Theorem 3.4. Thus, $\hat{\psi}_N(\lambda)$ has an appreciably higher rate of convergence--this at the expense of requiring a record of the sampling instants.

Comparison on the basis of mean integrated square error cannot be made since the kernel-type estimate (4.1) has not so far been shown to be consistent in this sense (see [2][5]).

The proofs of Theorems 3.3 and 3.4 indicate that the logarithmic convergence rates of the series estimates $\hat{\phi}_N(\lambda)$ are due to the exponential growth in n of $\text{var}[\hat{a}_n(N)]$ as given by the bound in Theorem 3.2. The discussion preceding Theorem 3.2 provides evidence for such a rapid growth. It remains an open question whether the convergence rates of the series estimate obtained here are the best possible.

Finally, we note that when the sampling process $\{t_n\}$ is not necessarily Poisson but "alias-free" in the sense of [1][10], an orthogonal series estimate of $\phi(\lambda)$ of the form (2.9) can be considered in a similar fashion. However, the analysis becomes more complex since the basis $\{g_n(t)\}_{n=1}^{\infty}$ is generated by the statistics of the point process $\{t_n\}$. (In the Poisson case, we have the Laguerre functions (2.2), whose properties are well known.)

5. Proofs.

Proof of Lemma 2.1. The following relationship for generalized Laguerre functions is easily verified.

$$(5.1) \quad t^{v-1} L_{n-1}^{v-1}(t) = \frac{d}{dt} \left[\frac{t^v}{(n+v-1)} L_{n-1}^v(t) \right].$$

By repeated substitution of (5.1) and integration by parts in the integral below, we have

$$(5.2) \quad \int_0^A C(t/2\beta) L_{n-1}(t) e^{-t/2} dt = \sum_{k=0}^{r-1} \frac{(-1)^k t^{k+1} L_{n-1}^{k+1}(t)}{n(n+1) \dots (n+k)} \frac{d^k}{dt^k} [C(t/2\beta) e^{-t/2}] \Big|_{0+}^A \\ + \frac{(-1)^r}{n(n+1) \dots (n+r-1)} \int_0^A \left\{ \frac{d^r}{dt^r} [C(t/2\beta) e^{-t/2}] \right\} t^r L_{n-1}^r(t) dt.$$

Each term in the sum vanishes at infinity by the hypothesis of Lemma 2.1.

Finally, by

$$\|t^{r/2} L_n^r(t) e^{-t/2}\|_{L_2(0,\infty)} = \left[\frac{(n+r)!}{n!}\right]^{1/2},$$

the dominated convergence theorem and the Cauchy-Schwarz inequality

$$|a_n| \leq A_1(r) \left[\frac{(n-1)!}{(n+r-1)!}\right]^{1/2} \leq \frac{A_1(r)}{n^{r/2}}.$$

Proof of Lemma 2.2. By (2.6) and Lemma 2.1,

$\sum_{n=1}^{\infty} |a_n| |G_n(\lambda)| \leq (2/\pi^2 \beta)^{1/2} \sum_{n=1}^{\infty} |a_n| < \infty$. Since $G_n(\lambda)$ is continuous, the series in (2.11) converges uniformly to an even continuous function, say, $\psi(\lambda)$. Since $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, $\psi(\lambda) \in L_2(-\infty, \infty)$ by the Riesz-Fischer theorem and hence $\int_{-\infty}^{\infty} [\phi(\lambda) - \psi(\lambda)] G_n(\lambda) d\lambda = 0$, $n = 1, 2, \dots$. Finally, since $\{G_n(\lambda)\}_{n=1}^{\infty}$ is complete in $L_2(-\infty, \infty)$ with respect to even functions, we have $\phi(\lambda) = \psi(\lambda)$ a.e. and the result follows by the continuity of $\phi(\lambda)$ and $\psi(\lambda)$.

Proof of Theorem 3.1. (i) is clear. For (ii), we have

$$\text{Var}[\hat{c}_n(N)] = \sum_{j=1}^4 T_{n,j}(N) - E^2[\hat{c}_n(N)]$$

where

$$T_{n,1}(N) = \frac{1}{N^2} \sum_{k, \ell=1}^{N-n} E[C(t_{k+n}-t_k)C(t_{\ell+n}-t_{\ell})],$$

$$T_{n,2}(N) = \frac{1}{N^2} \sum_{k, \ell=1}^{N-n} E[C(t_{\ell}-t_k)C(t_{\ell+n}-t_{k+n})],$$

$$T_{n,3}(N) = \frac{1}{N^2} \sum_{k, \ell=1}^{N-n} E[C(t_{\ell+n}-t_k)C(t_{k+n}-t_{\ell})],$$

$$T_{n,4}(N) = \frac{1}{N^2} \sum_{k, \ell=1}^{N-n} E[Q(t_{k+n}-t_k, t_{\ell}-t_k, t_{\ell+n}-t_k)].$$

By a bounding argument similar to the one employed in [5], it can be shown that

if $tC(t) \in L_2(-\infty, \infty)$ then $|T_{n,1}(N) - E^2[\hat{c}_n(N)]| \leq \frac{A_{2,1}}{N}$, if $C(t) \in L_1(-\infty, \infty)$ then $|T_{n,j}(N)| \leq \frac{A_{2,j}}{N}$, $j = 2, 3$, and if $Q(u_1, u_2, u_3)$ satisfies Assumption 3.1, then $|T_{n,4}(N)| \leq A_{2,4}/N$, where $A_{2,j}$, $j = 1, 2, 3, 4$ are constants independent of n and N . The result (ii) follows.

Proof of Theorem 3.2. (i) By (2.8), Theorem 3.1 and (2.4), $E[\hat{a}_n(N)] = a_n - (1/N) \sum_{k=1}^n k \theta_{n,k} c_k$. Thus the bias $b[\hat{a}_n(N)]$ is

$$(5.3) \quad b[\hat{a}_n(N)] = -(1/N) \int_0^\infty C(t) q_n(t) dt$$

where

$$q_n(t) = \sum_{k=1}^n k \theta_{n,k} f_k(t) = \sqrt{2\beta} \sum_{k=0}^{n-1} \frac{(k+1)}{k!} \binom{n-1}{k} (-2\beta t)^k e^{-\beta t}.$$

Using the properties of the Laguerre functions [7, p.299] it can be shown that $q_n(t) = n g_n(t) - (n-1)g_{n-1}(t)$ and the result follows by (5.3).

(ii) We have $\text{Var}^{1/2}[\hat{a}_n(N)] \leq \sum_{k=1}^n |\theta_{n,k}| \text{Var}^{1/2}[\hat{c}_k(N)]$ and the result follows by Theorem 3.1 and $\sum_{k=1}^n |\theta_{n,k}| = \sqrt{2/\beta} 3^{n-1}$.

Proof of Theorem 3.3. We have

$$E \int_{-\infty}^{\infty} |\hat{\phi}_N(\lambda) - \phi(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty} b^2[\hat{\phi}_N(\lambda)] d\lambda + \int_{-\infty}^{\infty} \text{Var}[\hat{\phi}_N(\lambda)] d\lambda$$

where by Theorem 3.2

$$(5.4) \quad \int_{-\infty}^{\infty} b^2[\hat{\phi}_N(\lambda)] d\lambda = \frac{1}{\pi} \sum_{n=1}^{\infty} \{a_n[1 - \gamma_n(N)] + \frac{\gamma_n(N)}{N} [n a_n - (n-1)a_{n-1}]\}^2,$$

and

$$(5.5) \quad \int_{-\infty}^{\infty} \text{Var}[\hat{\phi}_N(\lambda)] d\lambda = \frac{1}{\pi} \sum_{n=1}^{\infty} \gamma_n^2(N) \text{Var}[\hat{a}_n(N)].$$

The truncated sum in (5.5) at M , the integer part of $(b/\alpha) \ln N$, is $O(N^{-p})$, $p = 1 - (2b/\alpha) \ln 3$, by Theorem 3.2 and (3.2i); and by Theorem 3.2 and (3.2ii)

the tail sum is also $O(N^{-P})$. In fact,

$$(5.6) \quad \int_{-\infty}^{\infty} \text{Var}[\hat{\phi}_N(\lambda)] d\lambda \leq D_1 N^{-P}; \quad D_1 = \frac{A_3 k_1}{\pi(1-e^{-2(\alpha-\ln 3)})} + \frac{9A_3}{8\pi}.$$

Next (5.4) is bounded by $(1/\pi)(S_1 + S_2 + S_3)^2$ where

$$S_1^2 = \sum_{n=1}^{\infty} a_n^2 [1-\gamma_n(N)]^2, \quad S_2^2 = \frac{1}{N^2} \sum_{n=1}^{\infty} n^2 \gamma_n^2(N) a_n^2, \quad S_3^2 = \frac{1}{N^2} \sum_{n=1}^{\infty} n^2 \gamma_{n+1}^2(N) a_n^2.$$

S_1 is the dominant term in the integrated bias, for with M as before, we have

by (3.2i) and (3.2ii) and an argument similar to the one above that

$S_k^2 \leq (1/2)(b/\alpha)^2 \|C\|_{L_2}^2 (\ln N/N)^2 (1+o(1))$, $k = 2, 3$. Now for any integer $m > 1$, we have by (3.2i), (3.2iii) and Lemma 2.1 that

$$S_1^2 \leq (1/2) K_2^2 \|C\|_{L_2}^2 e^{2\alpha m} N^{-2b} + \frac{4A_1^2(r)}{r-1} \cdot \frac{1}{m^{r-1}}.$$

The optimal m which minimizes the right-hand side is then the solution of a transcendental equation and cannot be found explicitly. However, m is essentially logarithmic in N and upon choosing $m-1$ to be the integer part of $(b/2\alpha)\ln N$, we have

$$(5.7) \quad \int_{-\infty}^{\infty} b^2[\hat{\phi}_N(\lambda)] d\lambda \leq \frac{D_2}{(\ln N)^{r-1}} (1+o(1)), \quad D_2 = 4A_1^2(r) \frac{(2\alpha/b)^{r-1}}{(r-1)}.$$

The result follows by (5.6) and (5.7).

Proof of Theorem 3.4. By Lemma 2.2 and Theorem 3.2, the bias of $\hat{\phi}_N(\lambda)$ is

$$b[\hat{\phi}_N(\lambda)] = - \sum_{n=1}^{\infty} a_n [1-\gamma_n(N)] G_n(\lambda) - (1/N) \sum_{n=1}^{\infty} \gamma_n(N) [n a_n - (n-1) a_{n-1}] G_n(\lambda)$$

and by (2.6) $b[\hat{\phi}_N(\lambda)] \leq (2/\pi^2 \beta)^{1/2} (Z_1 + Z_2)$ uniformly in λ , where

$$Z_1 = \sum_{n=1}^{\infty} |a_n| [1-\gamma_n(N)],$$

$$Z_2 = (1/N) \sum_{n=1}^{\infty} n |a_n| [|\gamma_n(N)| + |\gamma_{n+1}(N)|].$$

Z_1 is the dominant term in the bias, for with M the integer part of $(b/\alpha)\ln N$, the truncated series at M in Z_2 is $O(\ln N/N)$ by (3.2i) and Lemma 2.1, and the tail sum is $O(N(\ln N)^{1-r/2})$ by (3.2iii). Next, for any integer $m>1$, we have by (3.2i)(3.2iii) and Lemma 2.1 that

$$Z_1 \leq \frac{k_2 A_1(r)r}{(r-2)} \cdot N^{-b} e^{\alpha m} + \frac{4A_1(r)}{(r-2)} \cdot m^{1-(r/2)},$$

and, by an argument similar to the one employed for S_1^2 of Theorem 3.3, we find

$$(5.8) \quad b[\hat{\phi}_N(\lambda)] \leq D_3 (\ln N)^{(2-r)/2} (1+o(1)); \quad D_3 = 4A_1(r)(b/\alpha)^{1-(r/2)}/(r-2).$$

For the variance we have by the Cauchy-Schwarz inequality and (2.6) that

$$\text{Var}[\hat{\phi}_N(\lambda)] \leq (2/\pi^2 \beta) \left[\sum_{n=1}^{\infty} |\gamma_n(N)| \text{var}^{1/2}[\hat{a}_n(N)] \right]^2.$$

Again, with M the integer part of $(b/\alpha)\ln N$, the truncated sum at M is $O(N^{-p/2})$, $p = 1 - \frac{2b}{\alpha} \ln 3$, by Theorem 3.2 and (3.2i); whereas the tail sum is $O(\sqrt{N})$ by Theorem 3.2 and (3.2ii). Hence uniformly in λ ,

$$(5.9) \quad \text{Var}[\hat{\phi}_N(\lambda)] = O(N^{-p}), \quad p = 1 - (2b/\alpha)\ln 3.$$

The result follows by (5.8) and (5.9).

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Unclassified, June 20, 1979

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Discrete-time spectral estimation of continuous-time processes -- The orthogonal series method		5. TYPE OF REPORT & PERIOD COVERED manuscript
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Elias Masry		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0652
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept of Applied Physics & Information Science University of California at San Diego La Jolla, CA 92093		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-289
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics and Probability Program Office of Naval Research Arlington, VA 22217		12. REPORT DATE April 1, 1979
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Distribution unlimited <div style="border: 1px solid black; padding: 5px; display: inline-block;">DISTRIBUTION STATEMENT A Approved for public release; Distribution Unlimited</div>		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Spectral estimation, orthogonal series estimates, convergence rates. <div style="text-align: center;">$-\infty$ ∞ $\phi(\lambda)$ $-\infty$</div>		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) <div style="display: flex;"><div style="margin-right: 20px;">λ ϕ</div><div>Let $X = \{X(t), -\infty < t < \infty\}$ be a stationary time series with spectral density function $\phi(\lambda)$. Let $\{t_i\}$ be a stationary Poisson point process on $(-\infty, \infty)$, independent of X. The existence of consistent estimates of $\phi(\lambda)$ based on the discrete-time observations $\{X(t_i)\}$, when the actual sampling instants $\{t_i\}$ are not known, has been an open question. Using an orthogonal series method, a class of spectral estimates is considered and its uniform and integrated uniform consistency in quadratic mean is established. The rates of convergence are determined and compared with those of the kernel-type estimates based on the observations $\{X(t_i), t_i\}$.</div></div>		

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